

ABSTRACT. This paper aims to help the development of new models of homotopy type theory, in particular with models that are based on realizability toposes. For this purpose it develops the foundations of an internal simplicial homotopy that does not rely on classical principles that are not valid in realizability toposes and related categories.

CONSTRUCTIVE SIMPLICIAL HOMOTOPY

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Introduction

This paper grew out of an attempt to build a *recursive realizability* model for *homotopy type theory* following the example of [Kapulkin, Lumsdaine, and Voevodsky 2012]. Their model is based on the category of simplicial sets. Simplicial homotopy is the homotopy of simplicial sets. It is equivalent to the homotopy of CW complexes which are a category of topological spaces [Goerss and Jardine 1999; Hovey 1999].

In the intended model types are homotopy types of certain simplicial objects in a *realizability topos* (see [Oosten 2008]). These toposes lack some of the structure of the category of sets underlying classical homotopy theory. In the internal logic the principle of the excluded middle and the axiom of choice can be invalid. They can lack infinite limits and colimits. We work around these issues in the following way.

1. Limit the class of cofibrations. In classical simplicial homotopy theory every monomorphism is a cofibration. We demand that certain properties of the monomorphisms are decidable so classical arguments remain valid.
2. Strengthen the lifting property. Fibrations come equipped with a *filler operator* that gives solutions for a family of basic lifting problems.
3. Build the homotopy category out of fibrant objects only, so we don't need fibrant replacements.

To avoid distracting peculiarities of the realizability topos this paper works with a generic Π -pretopos with a *natural number object*. That makes our simplicial homotopy predicative as well as constructive.

INTENDED MODEL The *category of assemblies* which is the category of $\neg\neg$ -separated objects of the effective topos up to equivalence, has several *strongly* complete internal categories that are not posets, in particular the *category of modest sets* [Hyland, Robinson, and Rosolini 1990; Oosten 2008; Rosolini 1990]. *Strongly complete* means that the externalization of the internal category is a complete fibred category. The category of assemblies is not exact, but the *ex/lex completion* [Carboni and Vitale 1998] preserves strongly complete internal categories. This exact completion is not the effective topos,

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but it is a kind of realizability topos and it is the intended ambient category \mathfrak{A} in this paper.

The topos with a complete internal category that is not a poset is interesting, because complete internal categories in Grothendieck toposes are necessarily posets. This fact is traditionally attributed to Peter Freyd.

In the ex/lex completion *modest fibrations* are fibrations whose underlying morphisms are families of modest sets or quotients of such families by modest families of equivalence relations—see [Hyland, Robinson, and Rosolini 1990; Oosten 2008; Rosolini 1990]. There is a *universal modest fibration* (see definition 6.1) which is a *univalent fibration* and hence a potential model of homotopy type theory.

CONCLUSION There are definitions of fibrations, cofibrations and their acyclic counterparts (definitions 2.1) in Π -pretoposes that make the category of Kan complexes a *model category* (theorem 2.11). Certain *universal fibrations* (definition 6.1) automatically live in the category of Kan complexes (theorem 6.2).

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1. Internal simplicial objects

Internal simplicial homotopy has a notion of simplicial object of \mathfrak{A} which does rely on external \mathfrak{A} -valued presheaves, but on internal structures.

1.1. DEFINITION. A simplicial object X of \mathfrak{A} is a triple $\langle \mathbf{U}X, \dim, \cdot \rangle$ where $\mathbf{U}X$ is an object of \mathfrak{A} , \dim is a morphism $\mathbf{U}X \rightarrow \mathbf{N}$. Here \mathbf{N} is the natural number object of \mathfrak{A} . The operator \cdot has the following properties. Let $\mathbf{Ar}(\Delta)$ be the object of non decreasing functions between decidable, finite and inhabited initial segments of the natural numbers. The domain $\text{dom}(\cdot)$ of \cdot is the following object.

$$\{\langle x, \xi \rangle : \mathbf{U}X \times \mathbf{Ar}(\Delta) \mid \dim(x) = \max(\text{cod}(\xi))\}$$

The codomain $\text{cod}(\cdot)$ of \cdot is $\mathbf{U}X$. The operator \cdot satisfies the following equations.

$$\begin{aligned} \dim(x \cdot \xi) &= \max(\text{dom}(\xi)) \\ x \cdot \text{id} &= x \\ x \cdot (\alpha \circ \beta) &= (x \cdot \alpha) \cdot \beta \end{aligned}$$

A morphism of simplicial objects $X \rightarrow Y$ is a morphism $f : \mathbf{U}X \rightarrow \mathbf{U}Y$ which commutes with \dim and \cdot . Together with simplicial objects, they form the category of simplicial objects and morphisms $\mathfrak{A}^{\Delta^{\text{op}}}$.

The category $\mathfrak{A}^{\Delta^{\text{op}}}$ is enriched over \mathfrak{A} . The object of morphisms $X \rightarrow Y$ in \mathfrak{A} is $\text{nat}(X, Y)$. It represents families of morphisms $X \rightarrow Y$, i.e. for each object I of \mathfrak{A} there is a natural bijection between morphisms $I \rightarrow \text{nat}(X, Y)$ and morphisms $I \times X \rightarrow Y$ which commute with dim , \cdot and their I -fold multiples.

1.2. EXAMPLE. A simple kind of simplicial object is the *discrete* simplicial object. There is one for each object I of \mathfrak{A} :

$$I_{\text{disc}} = \langle \mathbf{N} \times I, \lambda \langle n, i \rangle \mapsto n, \lambda \langle \langle n, i \rangle, \phi \rangle \mapsto i \rangle$$

Note the $\lambda x \mapsto f(x)$ notation for morphisms of \mathfrak{A} the paper uses.

This definition is consistent with the definition of initial presheaves in [Mac Lane and Moerdijk 1994]. The definition is valid because \mathfrak{A} is a Π -pretopos with a natural number object \mathbf{N} . Decidable, finite and inhabited initial segments of the natural numbers have a classifier.

$$\lambda \langle i, j \rangle \mapsto j : \{ \langle i, j \rangle : \mathbf{N} \times \mathbf{N} \mid i \leq j \} \rightarrow \mathbf{N}$$

The relation \leq is decidable. Comprehension on decidable predicates defines subobjects in \mathfrak{A} , because there classifiers of decidable subobjects $1 \rightarrow \text{bool} = 1 + 1$. Here 1 is a terminal object and $+$ a binary coproduct.

Local Cartesian closure means that there is an object A of morphisms between the initial segments. Local Cartesian closure also implies that there is a subobject of non decreasing morphisms $\mathbf{Ar}(\Delta)$.

$$\Pi f : A, i, j : \mathbf{N}, x : \{ y : 1 \mid i \leq j \} \mapsto \{ y : 1 \mid f(i) \leq f(j) \}$$

However, because \mathfrak{A} is also exact and extensive, it is a Heyting category, so the same subobject has a more familiar definition.

$$\mathbf{Ar}(\Delta) = \{ f : A \mid \forall i, j : \mathbf{N}. i \leq j \rightarrow f(i) \leq f(j) \}$$

1.3. DEFINITION. The object $\mathbf{Ar}(\Delta)$ is the object of arrows of the internal category of simplices Δ , whose object of objects $\mathbf{Ob}(\Delta)$ is \mathbf{N} . Composition is defined as ordinary function composition.

For completeness we recall the definition of Π -pretoposes below.

1.4. DEFINITION. That \mathfrak{A} is a Π -pretopos means all of the following.

1. The category \mathfrak{A} is locally Cartesian closed. This means that for each object X of \mathfrak{A} each slice \mathfrak{A}/X is Cartesian closed.
2. The category \mathfrak{A} is extensive, which means that it has finite coproducts and that $\mathfrak{A}/(X + Y)$ is equivalent to $(\mathfrak{A}/X) \times (\mathfrak{A}/Y)$ for each pair of objects X and Y of \mathfrak{A} .
3. The category \mathfrak{A} is exact in the sense of Barr. This means that all weak groupoids (see below) have coequalizers and that those coequalizers are stable under pullback.

A weak groupoid consists of a pair of objects X_0, X_1 and morphisms $r : X_0 \rightarrow X_1$ and $s, t : X_1 \rightarrow X_0$ such that $s \circ r = t \circ r = \text{id}_{X_0}$ such that for each $x, y, z : X_0$ if two of $\langle x, y \rangle$, $\langle x, z \rangle$ and $\langle y, z \rangle$ are in the image of $\langle s, t \rangle : X_1 \rightarrow X_0 \times X_0$, the third is too. A coequalizer for a weak groupoid is a coequalizer of s and t .

1.5. **EXAMPLE.** Every topos is a Π -pretopos so \mathfrak{A} can be any topos with a natural number object like the topos of sets.

This combination of properties implies the following.

1. The category \mathfrak{A} has all colimits and reindexing functors preserve them.
2. The category \mathfrak{A} is a *Heyting category*.

Technically this means that for each object X the poset of subobjects $\mathbf{Sub}(X)$ is a Heyting algebra and that for each $f : X \rightarrow Y$ the preimage map $f^* : \mathbf{Sub}(Y) \rightarrow \mathbf{Sub}(X)$ is a morphism of Heyting algebras. Moreover, for each morphism f , f^* has both adjoints—right adjoint \forall_f and left adjoint \exists_f —and those adjoints satisfy the *Beck-Chevalley condition* which says that quantification and substitution commute over pullback squares. If $f \circ g = h \circ k$ is a pullback square, then $f^* \circ \exists_h = \exists_g \circ k^*$.

Practically this means that \mathfrak{A} is a model for a constructive first order logic, hence that predicates define subobjects and that constructive theorems are valid in the internal language.

1.6. **REMARK.** We assume that all of the structure on \mathfrak{A} comes from functors. So several functors between categories related to \mathfrak{A} have right adjoints that give \mathfrak{A} its limits and exponentials and some functors have left adjoints that give \mathfrak{A} colimits. Definitions in terms of universal properties only define objects up to isomorphism. Our assumption ensures that there are functors that hit all the necessary isomorphism classes, when we only specify a functor up to isomorphism.

2. Model Category

This section defines a model structure on a category of $\mathfrak{A}^{\Delta^{\text{op}}}$.

2.1. **DEFINITION.** A model structure on an \mathcal{A} -enriched category \mathcal{C} consist of three sets of morphism: the fibrations F , the cofibrations C and the weak equivalences W which have the following properties.

- The set W satisfies 2-out-of-3 which means that for every pair of morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ of \mathcal{C} , if two out of f , g and $g \circ f$ are in W then all three are.
- The pairs of sets $(C, F \cap W)$ and $(C \cap W, F)$ are enriched factorization systems (see definition 2.2).

Categories with such a structure are model categories. Members of $C \cap W$ are called acyclic cofibrations and members of $F \cap W$ are called acyclic fibrations.

This notion of model structure is the ordinary one, except for the kind of factorization systems the morphisms in it form.

2.2. DEFINITION. An enriched factorization system is a pairs of set of morphisms (L, R) of an enriched category with the following properties.

- A morphism belong to L if and only if it has the left lifting property with respect all to members of R .
- A morphism belong to R if and only if it has the right lifting property with respect all to members of L .
- Every morphism factors as a member of R following a member of L .

A morphism $f : X \rightarrow Y$ of an \mathcal{A} -enriched category \mathcal{C} has the right lifting property with respect to a morphism $g : I \rightarrow J$ —and g has the left lifting property with respect to f —if the morphism $\langle f!, g^* \rangle = \langle \mathcal{C}(\text{id}_J, f), \mathcal{C}(g, \text{id}_X) \rangle$ which is the factorization of the span $\mathcal{C}(\text{id}_J, f), \mathcal{C}(g, \text{id}_X)$ through the pullback cone of $\mathcal{C}(\text{id}_I, f)$ and $\mathcal{C}(g, \text{id}_Y)$ is a split epimorphism.

$$\begin{array}{ccccc}
 & & \mathcal{C}(I, X) & & \\
 & \nearrow \mathcal{C}(g, \text{id}_X) & & \searrow \mathcal{C}(\text{id}_I, f) & \\
 \mathcal{C}(J, X) & \xrightarrow{\langle f!, g^* \rangle} \bullet & & & \mathcal{C}(I, Y) \\
 & \searrow \mathcal{C}(\text{id}_J, f) & & \nearrow \mathcal{C}(g, \text{id}_Y) & \\
 & & \mathcal{C}(J, Y) & &
 \end{array}$$

A section of $\langle f!, g^* \rangle$ is a filler operator.

In $\mathfrak{A}^{\Delta^{\text{op}}}$ we make the following selections.

2.3. DEFINITION. A morphism $f : X \rightarrow Y$ of category $\mathfrak{A}^{\Delta^{\text{op}}}$ is a fibration if it has the right lifting property with respect to the family of all horn inclusions, which is defined as follows.

For each $n \in \mathbb{N}$, the n -simplex $\Delta[n]$ is the internal simplicial set where $\mathbf{U}\Delta[n]$ is the object of all morphisms ϕ in Δ such that $\text{cod}(\phi) = n$, $\dim(\phi) = \text{dom}(\phi)$ and $\phi \cdot \xi = \phi \circ \xi$. Here \mathbb{N} is the set of natural numbers that exists outside of \mathfrak{A} . For each $k : [n]$ the horn $\Lambda_k[n]$ is the subobject of the non decreasing maps $[m] \rightarrow [n]$ that are not onto the set $[n] - \{k\}$.

$$\mathbf{U}\Lambda_k[n] = \{\phi : \mathbf{U}\Delta[n] \mid \exists i : [n] - \{k\}. \forall j : \text{dom}(\phi). \phi(i) \neq j\}$$

The horn inclusion is the monomorphism $\Lambda_k[n] \rightarrow \Delta[n]$. The family of horn inclusions is the sum of all horn inclusions $\Sigma k \leq n : \mathbb{N} \mapsto \Lambda_k[n] \rightarrow \Sigma k \leq n : \mathbb{N} \mapsto \Delta[n]$ that exists as a morphism of $\mathfrak{A}^{\Delta^{\text{op}}}$.

The lifting property that f satisfies is defined in a slice category of $\mathfrak{A}^{\Delta^{\text{op}}}$. Let $I = \{\langle n, k \rangle : \mathbb{N} \times \mathbb{N} \mid k \leq n\}$. The family of horn inclusions is a morphism in $\mathfrak{A}^{\Delta^{\text{op}}}/I_{\text{disc}}$ (see

example 1.2). So is the multiple $I_{\text{disc}} \times f : I_{\text{disc}} \times X \rightarrow I_{\text{disc}} \times Y$ of f . The morphism f is a fibration if $I_{\text{disc}} \times f$ has the right lifting property with respect to $\Sigma k \leq n : \mathbf{N} \mapsto \Lambda_k[n] \rightarrow \Sigma k \leq n : \mathbf{N} \mapsto \Delta[n]$.

2.4. EXAMPLE. If \mathfrak{A} is the topos of sets, then the morphisms that satisfy this definition are precisely the *Kan fibrations*. In that case the axiom of choice implies a filler operator exists.

2.5. DEFINITION. A monomorphism $f : X \rightarrow Y$ of category $\mathfrak{A}^{\Delta^{\text{op}}}$ is a cofibration if the subobject of faces of Y which are not in the image of X is decidable. A face of Y is a $y : \mathbf{U}Y$ such that if $\phi : [\text{dim}(y)] \rightarrow [\text{dim}(y)]$ in Δ and $y \cdot \phi = y$, then $\phi = \text{id}_{\text{dim}(y)}$. Elements of $\mathbf{U}Y$ which are not faces are called degenerate, so a face is a non degenerate element of $\mathbf{U}Y$. Hence f is a cofibration if there is a morphism $\psi : \mathbf{U}Y \rightarrow \mathbf{bool}$ such that $\psi(y) = \mathbf{false}$ if and only if y is degenerate or equal to $f(x)$ for some $x : \mathbf{U}X$.

2.6. EXAMPLE. In the category of simplicial sets all monomorphism are cofibrations. Decidability comes for free there.

2.7. DEFINITION. A weak equivalence is a composition of an acyclic fibration following an acyclic cofibration. Here, an acyclic fibration is a morphism that has the right lifting property with respect to all cofibrations; an acyclic cofibration is a cofibration has the right lifting property with respect to all fibrations.

The theorems in this paper should work in a Π -pretopos \mathfrak{A} that doesn't have infinite colimits. The small object argument uses such colimits to provide factorizations of morphisms and therefore we cannot use it here. To get a model structure we retreat to a subcategory of $\mathfrak{A}^{\Delta^{\text{op}}}$.

2.8. DEFINITION. A complex is a tuple $\langle \mathbf{U}X, \text{dim}, \cdot, \text{filler} \rangle$ where $\langle \mathbf{U}X, \text{dim}, \cdot \rangle$ is a simplicial object and filler is a filler operator (see definition 2.2) for the unique morphism $! : \langle \mathbf{U}X, \text{dim}, \cdot \rangle \rightarrow 1_{\text{disc}}$. A morphism of complexes is simply a morphism of simplicial objects. The category of complexes and morphisms of complexes is $\mathfrak{A}_f^{\Delta^{\text{op}}}$.

2.9. EXAMPLE. Each discrete simplicial object has a canonical filler operator, which means that $\lambda I : \mathfrak{A} \mapsto I_{\text{disc}}$ factors through the category of complexes. All simplices are complexes. Horns are not, however.

2.10. EXAMPLE. When \mathfrak{A} is the category of sets, then complexes are Kan complexes with a filler operator included.

2.11. THEOREM. [Model category] *With fibrations, weak equivalences and cofibrations defined as above the \mathfrak{A} -enriched category $\mathfrak{A}_f^{\Delta^{\text{op}}}$ is a model category.*

PROOF. Lemma 3.1 shows that cofibrations and acyclic fibrations form a weak factorization system. Lemma 5.1 tells the same thing about fibrations and acyclic cofibrations. Lemma 4.1 demonstrates that if two of f , g and $f \circ g$ are weak equivalences, then all three are. These three requirements define a model structure. ■

3. Cofibrations

This section show that the cofibrations from definition 2.5 are part of an enriched factorization system (see definition 2.2) on $\mathfrak{A}^{\Delta^{\text{op}}}$ —not just on the subcategory of complexes.

3.1. LEMMA. *Cofibrations and acyclic fibrations form an enriched factorization system on $\mathfrak{A}^{\Delta^{\text{op}}}$.*

PROOF. Proposition 3.4 shows that every morphism factors as a *contractible morphism* (see definition 3.2) following a cofibration. Acyclic fibrations are examples of contractible morphisms (see example 3.3). Lemma 3.5 shows that contractible morphisms are acyclic fibrations. Hence contractible morphisms and acyclic fibrations are the same class of morphism and this class form the right class of a factorization system with cofibrations on the left (see definition 2.2). ■

3.2. DEFINITION. *A morphism $f : X \rightarrow Y$ is contractible if it has the internal left lifting property with respect to the family of cycle inclusions, which is defined as follows.*

The cycle $\partial\Delta[n]$ is the subobject of non surjective functions of $\Delta[n]$. The dim and \cdot are the same and the bases are related as follows:

$$\mathbf{U}(\partial\Delta[n]) = \{\phi : \mathbf{U}(\Delta[n]) \mid \exists i : [n]. \forall j : \text{dom}(\phi). \phi(i) \neq j\}$$

The cycle inclusion is the monomorphism $\partial\Delta[n] \rightarrow \Delta[n]$. The family of cycle inclusions is the sum of all cycle inclusions $\Sigma n : \mathbf{N} \mapsto \partial\Delta[n] \rightarrow \Sigma k \leq n : \mathbf{N} \mapsto \Delta[n]$ that exists as a morphism of $\mathfrak{A}^{\Delta^{\text{op}}}$. Note that $\Sigma x : y \mapsto z(x)$ denotes the indexed coproduct of the y -indexed family z .

Definition 2.3 explains how lifting properties with respect to families work.

3.3. EXAMPLE. Each cycle inclusion $\partial\Delta[n] \rightarrow \Delta[n]$ is a cofibration because $\text{id}_{[n]} : [n] \rightarrow [n]$ is the unique face of $\Delta[n]$ not in $\partial\Delta[n]$ and equality with id is decidable in Δ . Therefore every acyclic fibration is a contractible morphism.

3.4. PROPOSITION. *Every morphism factors as a contractible morphism following a cofibration.*

PROOF. Let Δ^+ be the subcategory of monomorphisms and Δ^- that of epimorphisms of Δ (see definition 1.3). For each morphism ϕ of Δ let $m(\phi)$ be the monic and $e(\phi)$ the epic factors.

The first step is to cover Y with another simplicial set LY where degeneracy is decidable. Let $LY[n] = \Sigma i : [n] \mapsto Y[i] \times \Delta^-(n, i)$ and for $\phi : [m] \rightarrow [n]$ of Δ and $\langle \epsilon, y \rangle : LY[n]$ let the following equation hold.

$$\langle \epsilon, y \rangle \cdot \phi = \langle e(\epsilon \circ \phi), y \cdot m(\epsilon \circ \phi) \rangle$$

Let $l_Y : LY \rightarrow Y$ satisfy $l_Y \langle \epsilon, y \rangle = y \cdot \epsilon$.

The second step glues simplices of X and LY together into new ones. The pair (f, l_Y) stand for the morphism $X + LY \rightarrow Y$ that satisfies $(f, l_Y)(x) = f(x)$ for $x : X$ and

$(f, l_Y)(y) = l_Y(y)$ for $y : LY$. Let $Z[n] \subseteq \Pi i \in [n] \mapsto (X[i] + LY[i])^{\Delta^+(i,n)}$ consist of elements z which satisfy the following conditions for all $\alpha : [j] \rightarrow [k]$ and $\beta : [i] \rightarrow [j]$ in Δ^+ .

1. $(f, l_Y)(z(\alpha \circ \beta)) = (f, l_Y)(z(\alpha)) \cdot \beta$;
2. if $z(\alpha) : X$ then $z(\alpha \circ \beta) : X$;
3. if $z(\alpha) = \langle \epsilon, y \rangle : LY$ and $\epsilon \circ \beta$ is not monic, then $z(\alpha \circ \beta) : LY$ and $z(\alpha \circ \beta)_0$ factors through $\epsilon \circ \beta$.

For $\phi : [m] \rightarrow [n]$ and $z : Z[n]$, let $z \cdot \phi$ satisfy the following equation for all α of Δ^+ .

$$(z \cdot \phi)(\alpha) = z(m(\phi \circ \alpha)) \cdot e(\phi \circ \alpha)$$

By these definitions, $Z = (\Sigma n : \mathbf{N} \mapsto Z[n], \lambda \langle n, z \rangle \mapsto n, \cdot)$ is a simplicial object.

There is a morphism $g : X \rightarrow Z$ which satisfies $g(x)(\alpha) = x \cdot \alpha$. It is a cofibration because a $z : \mathbf{U}Z$ is nondegenerate and outside of the image of g if and only if $z(\text{id}_{[\dim(z)]}) = \langle \text{id}_{[\dim(z)]}, y \rangle$ for some $y : Y[\dim(z)]$. The definition of LY ensures that elements with this property are nondegenerate while condition 3 ensures that if $z(\text{id})$ is degenerate, then so is all of z .

There is a morphism $h : Z \rightarrow Y$ which satisfies $h(z) = (f, l_Y)(z(\text{id}_{[\dim(z)]}))$. This morphism is contractible thanks to the following filler construction. Let $a : \partial\Delta[n] \rightarrow Z$ and $b : \Delta[n] \rightarrow Y$ satisfy $h \circ a = b \circ k$ where $k : \partial\Delta[n] \rightarrow \Delta[n]$ is the cycle inclusion. Let $c : \Delta[n] \rightarrow Z$ satisfy $c(\phi)(\mu) = \langle \phi, b(\mu) \rangle$ if ϕ is an epimorphism and $c(\phi)(\mu) = a(\phi)(\mu)$ otherwise. Now $c \circ k = a$ and $h \circ c = b$ so c is a filler. There is a morphism in $\mathfrak{A}^{\Delta^{\text{op}}}$ that maps each commutative square $h \circ a = b \circ k$ to this filler c because $\mathfrak{A}^{\Delta^{\text{op}}}$ is locally Cartesian closed and being an epimorphism is a decidable property of morphisms of Δ . ■

3.5. LEMMA. *Contractible morphisms are acyclic fibrations.*

PROOF. The class of morphisms that have the left lifting property with respect to contractible morphism is saturated (see lemma 3.7). The rest proof shows that all cofibrations are in this class. That implies that all contractible morphisms have the right lifting property with respect to all cofibrations and therefore are acyclic fibrations.

Suppose $f : X \rightarrow Y$ is a cofibration and S is the family of faces outside the image of f . For each $i : \mathbf{N}$ let Y_i be the union of X with all faces of Y of dimension strictly smaller than i . In particular $Y_0 = X$.

Every $y : \mathbf{U}(Y)$ is in Y_i for some $i \leq \dim(y)$. If a $y : \mathbf{U}Y$ is degenerate, then one can search the monomorphisms $[n] \rightarrow \dim(y)$ in Δ for the greatest μ such that $y \cdot \mu$ is a face. For this reason, the inclusion $Y_i \rightarrow \mathbf{N}^*Y$ is a transfinite composition (see definition 3.6).

For each $i : \mathbf{N}$ let S_i be the object of n -dimensional faces in S . Each $s : S_i$ induces a monomorphism $s' : \Delta[i] \rightarrow Y_{i+1}$. Since Y_i has all simplices of dimension $i - 1$, the intersection of Y_i and s' is $\partial\Delta[i]$. Thus the inclusion $Y_i \rightarrow Y_{i+1}$ is the pushout of S_{i+1} copies of $\partial\Delta[i] \rightarrow \Delta[i]$. ■

3.6. DEFINITION. Let \mathcal{A} be a Π -pretopos with a natural number object \mathbf{N} . The successor morphism $s : \mathbf{N} \rightarrow \mathbf{N}$ induces an endofunctor s^* of \mathcal{A}/\mathbf{N} . A cochain is a morphism $f : X \rightarrow s^*(X)$ in \mathcal{A}/\mathbf{N} . A morphism of cochains $(X, f) \rightarrow (Y, g)$ is a morphism $h : X \rightarrow Y$ such that $s^*(h) \circ f = g \circ h$.

For every object Y of \mathcal{A} there is a constant family $\mathbf{N}^*(Y)$ and a canonical isomorphism $\mathbf{N}^*(Y) \simeq s^*(\mathbf{N}^*(Y))$ which makes $\mathbf{N}^*(Y)$ a cochain. The transfinite composition of a cochain (X, g) is a cochain morphism $\omega(g) : (X, f) \rightarrow \mathbf{N}^*Y$ such that for every other morphism $k : (X, f) \rightarrow \mathbf{N}^*Z$ there is a unique morphism $l : Z \rightarrow Y$ such that $s^*(l) \circ \omega(g) = k$.

3.7. LEMMA. The class of morphisms L that has the left lifting property with respect to fibrations is closed under pushouts, coproducts indexed over objects of \mathfrak{A} and transfinite compositions.

PROOF. In each case the construction induces an operation on the split epimorphism in the diagram of the lifting property. These operations happen to preserve split epimorphisms.

Suppose $h : I' \rightarrow J'$ is a pushout of $g : I \rightarrow J$ where g has the left lifting property with respect to $f : X \rightarrow Y$. Because the functors $\text{nat}(-, X)$ and $\text{nat}(-, Y) : (\mathfrak{A}^{\Delta^{\text{op}}})^{\text{op}} \rightarrow \mathfrak{A}$ send pushouts to pullbacks, $\langle f_!, h^* \rangle$ is a pullback of $\langle f_!, g^* \rangle$ and the former is a split epimorphism because the latter is.

Suppose $h : (\Pi i : I \mapsto \text{nat}(X(i), Y(i)))$ represents a family of morphisms of that share the left lifting property. That means $\langle (I_{\text{disc}}^*(f))_!, h^* \rangle$ is a split epimorphism and therefore so is its transpose $\langle f_!, \Pi_I(h)^* \rangle$, where $\Pi_I(h) : \text{nat}(\Sigma i : I \mapsto X(i), \Sigma i : I \mapsto Y(i))$ is the indexed coproduct of the family h .

Finally, suppose that a cochain (Z, h) has the left lifting property. That means $\langle \mathbf{N}_{\text{disc}}^*(f)_!, h^* \rangle$ is a split epimorphism. For the transfinite composition $\omega(h)$, $\langle f_!, \omega(h)^* \rangle$ is therefore also split. ■

4. Weak Equivalences

This section demonstrates that the weak equivalences of definition 2.7 satisfy the two out of three property (see definition 2.1).

4.1. LEMMA. [2-out-of-3] Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be morphisms of $\mathfrak{A}_f^{\Delta^{\text{op}}}$. If any two of f, g or $g \circ f$ are weak equivalences, then all three are.

PROOF. Weak equivalences are closed under composition by lemma 4.2.

Let g and $g \circ f$ be arbitrary weak equivalences. The morphism f factors as an acyclic fibration $h : W \rightarrow Y$ following a cofibration $k : X \rightarrow W$ by proposition 5.5. Because weak equivalences are closed under composition (see 4.2), $g \circ h$ is a weak equivalence. The morphism k is acyclic for the following reasons. Factor both $g \circ f$ and $g \circ h$ as acyclic fibrations following acyclic cofibrations, so $g \circ f = a \circ b$ and $g \circ h = c \circ d$. The lifting properties induce a morphism l such that $l \circ b = d \circ k$ and $c \circ l = a$. Lemma 4.6 says that l is a weak equivalence because a and c are acyclic fibrations. Because of closure under

composition, the morphism $l \circ b = d \circ k$ is both a weak equivalence and a cofibration and hence an acyclic cofibration. Since d and $d \circ k$ are acyclic cofibrations, so is k by lemma 4.4.

$$\begin{array}{ccccc}
 & & f & & \\
 X & \xrightarrow{k} & W & \xrightarrow{h} & Y \\
 \downarrow b & & \downarrow d & & \downarrow g \\
 \bullet & \xrightarrow{\quad l \quad} & \bullet & \xrightarrow{c} & Z \\
 & & a & &
 \end{array}$$

Since $f = h \circ k$, f is a weak equivalence.

The case where f and $g \circ f$ are weak equivalences is dual to the case above and the reasoning is the same. Acyclic cofibrations satisfy lemma 4.7 where acyclic fibrations satisfy lemma 4.6. Acyclic fibrations satisfy lemma 4.5 where acyclic cofibrations satisfy lemma 4.4.

This means that weak equivalences indeed satisfy 2-out-of-3. ■

4.2. LEMMA. *Weak equivalences are closed under composition.*

PROOF. Compositions of acyclic fibrations are acyclic fibrations and the same holds for acyclic cofibrations. All compositions of weak equivalences are weak equivalences, if $g \circ f$ factors as an acyclic fibration following an acyclic cofibration for each acyclic cofibration g and acyclic fibration f .

By proposition 3.4 $g \circ f = h \circ k$ for some acyclic fibration $h : W \rightarrow Z$ and a cofibration $k : X \rightarrow W$. Let g' be the left inverse of g . Since $f \circ \text{id} = g' \circ g \circ f = (g' \circ h) \circ k$ there is a morphism k' such that $f \circ k' = g' \circ h$ and $k' \circ k = \text{id}$, so k has its own left inverse.

$$\begin{array}{ccccc}
 & & \text{id} & & \\
 X & \xrightarrow{k} & W & \xrightarrow{k'} & X \\
 \downarrow f & & \downarrow h & & \downarrow f \\
 Y & \xrightarrow{g} & Z & \xrightarrow{g'} & Y \\
 & & \text{id} & &
 \end{array}$$

Let ϕ be a homotopy between id_Z and $g \circ g'$ such that $\phi \circ (\text{id} \times g) = g \circ \pi_1$.

There is a homotopy χ between id_W and $k \circ k'$ by lemma 4.3.

$$\begin{aligned}
 (\text{id}, k \circ k') \circ (k + k) &= (k, k) = k \circ \pi_1 \circ (c \times \text{id}_X) \\
 h \circ (k \circ \pi_1) &= (\phi \circ (\text{id}_W \times h)) \circ (\text{id}_{\Delta[1]} \times k) \\
 h \circ (\text{id}, k \circ k') &= (\phi \circ (\text{id}_W \times h)) \circ (c \times \text{id}_W)
 \end{aligned}$$

$$\begin{array}{ccccc}
 X + X & \xrightarrow{k+k} & W + W & \xrightarrow{(\text{id}, k \circ k')} & W \\
 \downarrow c \times \text{id} & & \downarrow c \times \text{id} & \nearrow \chi & \downarrow h \\
 \Delta[1] \times X & \xrightarrow{\text{id} \times k} & \Delta[1] \times W & \xrightarrow{\phi \circ (\text{id} \times h)} & Z
 \end{array}$$

Because the homotopy satisfies $\chi \circ (\text{id}_{\Delta[1]} \times k) = k \circ \pi_1$, k is a weakly invertible cofibration by definition 5.2 and an acyclic cofibration by lemma 5.3. ■

4.3. LEMMA. [Triple lifting property] *Let $f : A \rightarrow B$ and $g : C \rightarrow D$ be cofibrations and let $h : X \rightarrow Y$ be a fibration. Let $a : A \times D \rightarrow X$, $b : B \times C \rightarrow X$ and $c : B \times D \rightarrow Y$ satisfy $a \circ (\text{id}_A \times g) = b \circ (f \times \text{id}_C)$, $h \circ a = c \circ (f \times \text{id}_D)$ and $h \circ b = c \circ (\text{id}_B \times g)$. If one of f , g or h is acyclic, then there is a $d : B \times D \rightarrow X$ such that $d \circ (f \times \text{id}_D) = a$, $d \circ (\text{id}_B \times g) = b$ and $h \circ d = c$.*

$$\begin{array}{ccccc}
 A \times C & \xrightarrow{f \times \text{id}} & B \times C & \xrightarrow{b} & X \\
 \text{id} \times g \downarrow & & \downarrow \text{id} \times g & \nearrow a & \downarrow h \\
 A \times D & \xrightarrow{f \times \text{id}} & B \times D & \xrightarrow{c} & Y \\
 & & & \nearrow d &
 \end{array}$$

PROOF. The general case reduces to the cases where f and g are cycle or horn inclusions, because the lemma is equivalent to the statement that the following maps are (acyclic) fibrations and because acyclic cofibrations are contractible (see definition 3.2).

$$\begin{aligned}
 \langle f \rightarrow h \rangle &= \langle h^{\text{id}_B}, \text{id}_X^f \rangle : X^B \rightarrow Y^B \times_{Y^A} X^A \\
 \langle g \rightarrow h \rangle &= \langle h^{\text{id}_D}, \text{id}_X^g \rangle : X^D \rightarrow Y^D \times_{Y^C} X^C
 \end{aligned}$$

In the case where h is an acyclic fibration the morphism $\langle g \rightarrow h \rangle$ is an acyclic fibration if it has the lifting property for the family of cycle inclusions $k_i : \partial\Delta[i] \rightarrow \Delta[i]$ for $i : \mathbf{N}$, because acyclic fibrations are contractible morphisms (see lemma 3.5). In turn $\langle k_n \rightarrow h \rangle$ are acyclic cofibrations if the triple lifting property holds in the cases where $a = k_i$ and $b = k_j$ for all $i, j : \mathbf{N}$.

For the cases where f or g are acyclic the same reduction takes us to products of cycle and horn inclusions. If f is acyclic, the problem of lifting f against $g \rightarrow h$ reduces to the problem of lifting horn inclusions. The problem of lifting a possibly not acyclic g against $k \rightarrow h$, where k is a horn inclusion, reduces to the problem of lifting cycle inclusions by lemma 3.5.

In the cases where f and g are horn or cycle inclusions it is easy to prove that their pushout products $g \diamond f$ are compositions of pushouts of sums of horns and cycles and therefore have the left lifting properties with respect to (acyclic) fibrations (see lemma 3.7).

$$\begin{array}{ccccc}
 & & X \times Y & & \\
 f \times \text{id} \nearrow & & \searrow \text{id} \times g & & \\
 W \times Y & & \bullet & \xrightarrow{g \diamond f} & X \times Z \\
 \text{id} \times g \searrow & & \nearrow f \times \text{id} & & \\
 & & W \times Z & &
 \end{array}$$

Therefore the lemma holds for every pair of cofibrations f , g as long as one of f , g and h is acyclic. ■

4.4. LEMMA. *If $f : X \rightarrow Y$, $g : Y \rightarrow Z$ and if g and $g \circ f$ are acyclic cofibrations, then f is an acyclic cofibration*

PROOF. Let $k : A \rightarrow B$ be a fibration and let $a : X \rightarrow A$ and $b : Y \rightarrow B$ satisfy $k \circ a = b \circ f$. Because B is a complex, there is a $b' : Z \rightarrow B$ such that $b' \circ g = b$. Lifting properties also imply that there is an $a' : Z \rightarrow A$ such that $a' \circ g \circ f = a$ and $k \circ a' = b'$. So $b' \circ g$ is a filler for $k \circ a = b \circ f$. By generalization this construction lifts f against all fibrations and that makes f an acyclic cofibration.

$$\begin{array}{ccc}
 X & \xrightarrow{a} & A \\
 f \downarrow & \nearrow a' & \downarrow k \\
 Y & \xrightarrow{b} & B \\
 g \downarrow & \nearrow b' & \\
 Z & &
 \end{array}$$

■

4.5. LEMMA. *If $f : X \rightarrow Y$, $g : Y \rightarrow Z$ and if f and $g \circ f$ are acyclic fibrations, then g is an acyclic fibration.*

PROOF. This is nearly the dual of lemma 4.4 and dual reasoning gives g the right lifting property for all cycle inclusions which makes g contractible (see definition 3.2) and an acyclic cofibration by lemma 3.5.

$$\begin{array}{ccc}
 & & X \\
 & \nearrow a' & \downarrow \\
 \partial\Delta[n] & \xrightarrow{a} & Y \\
 k \downarrow & \nearrow b' & \downarrow \\
 \Delta[n] & \xrightarrow{b} & Z
 \end{array}$$

The reason the dual reasoning works is that for the cycles $\partial\Delta[n]$ from definition 3.2 the unique morphism $! : 0 \rightarrow \partial\Delta[n]$ is a cofibration. Here $0 = 0_{\text{disc}}$ is the initial object of $\mathfrak{A}^{\Delta^{\text{op}}}$. A member of $\mathbf{U}(\partial\Delta[n])$ is a face if it is an identity morphism, and this is a decidable property of morphism in Δ . No faces are in the image of $!$. ■

4.6. LEMMA. *If $f : X \rightarrow Y$, $g : Y \rightarrow Z$ and if g and $g \circ f$ are acyclic fibrations, then f is a weak equivalence.*

PROOF. By proposition 3.4, f factors as an acyclic fibration $h : W \rightarrow Y$ following a cofibration $k : X \rightarrow W$. Because $(g \circ f) \circ \text{id} = (g \circ h) \circ k$ and $g \circ f$ is an acyclic fibration,

k has a left inverse $k' : X \rightarrow W$ which satisfies $g \circ f \circ k' = g \circ h$.

$$\begin{array}{ccc} X & \xrightarrow{\text{id}} & X \\ k \downarrow & \nearrow k' & \downarrow g \circ f \\ W & \xrightarrow{g \circ h} & Z \end{array}$$

Let $c : 1 + 1 \rightarrow \Delta[1]$ be the same cycle as above. There is homotopy ϕ between id_W and $k \circ k'$ by lemma 4.3 and the following equations.

$$\begin{aligned} (\text{id}, k \circ k') \circ (k + k) &= (k, k) = (k \circ \pi_1) \circ (c \times \text{id}_X) \\ (g \circ h) \circ (k \circ \pi_1) &= (g \circ h \circ \pi_1) \circ (\text{id} \times k) \\ (g \circ h) \circ (\text{id}, k \circ k') &= (g \circ h \circ \pi_1) \circ (c \times \text{id}_W) \end{aligned}$$

$$\begin{array}{ccccc} X + X & \xrightarrow{k+k} & W + W & \xrightarrow{(\text{id}, k \circ k')} & W \\ c \times \text{id} \downarrow & \nearrow k \circ \pi_1 & c \times \text{id} \downarrow & \nearrow \phi & \downarrow g \circ h \\ \Delta[1] \times X & \xrightarrow{\text{id} \times k} & \Delta[1] \times W & \xrightarrow{g \circ h \circ \pi_1} & Z \end{array}$$

Because $\phi \circ (\text{id}_{\Delta[1]} \times k) = k \circ \pi_1$, definition 5.2 says that k is a weakly invertible cofibration and lemma 5.3 says that k is an acyclic cofibration. Therefore f is a weak equivalence. ■

4.7. LEMMA. *If $f : X \rightarrow Y$, $g : Y \rightarrow Z$ and if f and $g \circ f$ are acyclic cofibrations, then g is a weak equivalence.*

PROOF. This lemma is dual to lemma 4.6. The morphism g factors as a fibration $h : W \rightarrow Z$ following an acyclic cofibration $k : Y \rightarrow W$ by lemma 5.5. The dual of the proof of lemma 4.6 doesn't show that h is an acyclic fibration directly, but it does show that h has a right inverse $h' : Z \rightarrow W$ and that there is a homotopy $\phi : \Delta[1] \times W \rightarrow W$ between id_W and $h' \circ h$.

Let $a : I \rightarrow J$ be an arbitrary cofibration and let $i : I \rightarrow W$ and $j : J \rightarrow Z$ satisfy $j \circ a = h \circ i$. Lemma 4.3 deforms $h' \circ j$ into a filler. Let c_i be the morphisms $1 \rightarrow \Delta[0]$.

$$\begin{aligned} \phi \circ (\text{id} \times i) \circ (c_1 \times \text{id}) &= h' \circ h \circ i = h' \circ j \circ a \\ h \circ h' \circ j &= j = j \circ \pi_1 \circ (c_1 \times \text{id}) \\ j \circ \pi_1 \circ (\text{id} \times a) &= j \circ a \circ \pi_1 = h \circ i \circ \pi_1 \\ h \circ \phi \circ (\text{id} \times i) &= h \circ \pi_1 \circ (\text{id} \times i) = h \circ i \circ \pi_1 \end{aligned}$$

$$\begin{array}{ccccc} I & \xrightarrow{a} & J & \xrightarrow{h' \circ j} & W \\ c_1 \times \text{id} \downarrow & \nearrow \phi \circ (\text{id} \times i) & c_1 \times \text{id} \downarrow & \nearrow \psi & \downarrow h \\ \Delta[1] \times I & \xrightarrow{\text{id} \times a} & \Delta[1] \times J & \xrightarrow{j \circ \pi_1} & Z \end{array}$$

The filler is $\psi \circ c_0$. By generalization, h is an acyclic fibration and f is a weak equivalence. ■

5. Fibrations

This section show that fibrations are part of a factorization system as well.

5.1. LEMMA. *Acyclic cofibrations and fibrations form an enriched factorization system on $\mathfrak{A}_f^{\Delta^{\text{op}}}$.*

PROOF. Lemma 5.3 shows that weakly invertible cofibrations (see definition 5.2) are acyclic cofibrations. Lemma 5.4 shows the converse. Proposition 5.5 shows that every morphism factors as a fibration following a weakly invertible cofibration. ■

5.2. DEFINITION. *In the subcategory $\mathfrak{A}_f^{\Delta^{\text{op}}}$ of $\mathfrak{A}^{\Delta^{\text{op}}}$ a cofibration $f : X \rightarrow Y$ is weakly invertible if there is a $g : Y \rightarrow X$ such that $g \circ f = \text{id}_X$ and an $h : \Delta[1] \times Y \rightarrow Y$ such that $h \circ k = (\text{id}_Y, f \circ g)$ if $k : 1 + 1 \rightarrow \Delta[1]$ is the cycle inclusion and $h \circ (\text{id}_{\Delta[1]} \times f) = f \circ \pi_1$.*

$$\begin{array}{ccc}
 Y + Y & & \\
 k \times \text{id} \downarrow & \searrow (\text{id}, f \circ g) & \\
 \Delta[1] \times Y & \xrightarrow{h} & Y \\
 \text{id} \times f \uparrow & & \uparrow f \\
 \Delta[1] \times X & \xrightarrow{\pi_1} & X
 \end{array}$$

5.3. LEMMA. *Weakly invertible cofibrations are acyclic cofibrations.*

PROOF. Let $f : X \rightarrow Y$ be a weakly invertible cofibration, with inverse $g : Y \rightarrow X$ and homotopy $h : \Delta[1] \times Y \rightarrow Y$. Let $k : A \rightarrow B$ be a fibration and let $a : X \rightarrow A$ and $b : Y \rightarrow B$ satisfy $k \circ a = b \circ f$. The composition $a \circ g : Y \rightarrow A$ is a filler up to homotopy, which can be transported along the homotopy h to become a proper filler. Let d_0 be the first horn inclusion $1 \rightarrow \Delta[1]$. By lemma 4.3, there is a $d : \Delta[1] \times Y \rightarrow A$ such that $k \circ c = b \circ h$, $c \circ (\text{id}_{\Delta[1]} \times f) = a \circ \pi_1$ and $c \circ (d_0 \times \text{id}_Y) = a \circ g$. Lemma 4.3 applies because the following equations hold.

$$\begin{aligned}
 (a \circ \pi_1) \circ (\Delta(\delta_1^1) \times \text{id}_X) &= a = (a \circ g) \circ f \\
 (b \circ h) \circ (\Delta(\delta_1^1) \times \text{id}_Y) &= b \circ f \circ g = k \circ (a \circ g) \\
 (b \circ h) \circ (\text{id}_{\Delta[1]} \times f) &= b \circ f \circ \pi_1 = k \circ (a \circ \pi_1)
 \end{aligned}$$

$$\begin{array}{ccccc}
 X & \xrightarrow{d_0 \times \text{id}} & \Delta[1] \times X & \xrightarrow{a \circ \pi_1} & A \\
 f \downarrow & & \downarrow \text{id} \times f & \nearrow c & \downarrow k \\
 Y & \xrightarrow{d_0 \times \text{id}} & \Delta[1] \times Y & \xrightarrow{b \circ h} & B
 \end{array}$$

Let d_1 be the other horn inclusion $1 \rightarrow \Delta[1]$. Let $e = c \circ (d_1 \times \text{id}_Y)$, so $k \circ e = b \circ h \circ (d_1 \times \text{id}_Y) = b$ and $e \circ f = c \circ (\text{id}_{\Delta[1]} \times f) \circ (d_1 \times \text{id}_X) = a$. Then e is a filler for $b \circ f = k \circ a$. By generalization f is a weakly invertible cofibration. ■

5.4. LEMMA. *All acyclic cofibrations in $\mathfrak{A}_f^{\Delta^{\text{op}}}$ are weakly invertible.*

PROOF. An acyclic fibration is a fibration, because the right lifting property for cofibrations implies the right lifting property for horn inclusions. Therefore every acyclic cofibration is a cofibration.

Let $f : X \rightarrow Y$ be a morphism with the left lifting property for all fibrations in $\mathfrak{A}_f^{\Delta^{\text{op}}}$. There is a morphism $g : Y \rightarrow X$ such that $g \circ f = \text{id}_X$ because X is fibrant.

$$\begin{array}{ccc} X & \xrightarrow{\text{id}} & X \\ f \downarrow & \nearrow h & \downarrow ! \\ Y & \xrightarrow{\quad} & 1 \\ & \downarrow ! & \end{array}$$

There is a morphism $h : \Delta[1] \times Y \rightarrow Y$ such that $h \circ k = (\text{id}_Y, f \circ g)$ if $k : 1 + 1 \rightarrow \Delta[1]$ is the cycle inclusion and $h \circ (\text{id}_{\Delta[1]} \times f) = f \circ \pi_1$ because Y is a complex and because of lemma 4.3.

$$\begin{array}{ccccc} X + X & \xrightarrow{k \times \text{id}} & \Delta[1] \times X & \xrightarrow{f \circ \pi_0} & Y \\ f+f \downarrow & \nearrow (\text{id}, g) & \downarrow & \nearrow h & \downarrow ! \\ Y + Y & \xrightarrow{k \times \text{id}} & \Delta[1] \times Y & \xrightarrow{\quad} & 1 \\ & & \downarrow ! & & \end{array}$$

Therefore f is a weakly invertible cofibration by definition 5.2. ■

5.5. PROPOSITION. *Every morphism $f : X \rightarrow Y$ of $\mathfrak{A}_f^{\Delta^{\text{op}}}$ factors as a fibration following a weakly invertible cofibration.*

PROOF. There is a simple factorization $f = p_0 \circ r$ where $p_0 : Y/f \rightarrow Y$ and $r : X \rightarrow Y/f$ and where

$$\begin{aligned} Y/f &= \{(x, p) : X \times Y^{\Delta[1]} \mid f(x) = p(\lambda * \mapsto 1)\} \\ p_0 \langle x, p \rangle &= p(\lambda * \mapsto 0) \\ r(x) &= \langle x, \lambda * \mapsto f(x) \rangle \end{aligned}$$

The morphism r is not necessarily a cofibration, but factors as an acyclic fibration $g : W \rightarrow Y/f$ following a cofibration $h : X \rightarrow W$ by proposition 3.4. The morphism h is a weakly invertible cofibration and the composition $p_0 \circ g$ is a fibration for the following reasons.

There is a morphism $p_1 : Y/f \rightarrow X$ that satisfies $p_1 \langle x, p \rangle = x$. Let $h' = p_1 \circ h$, so $h' \circ h = \text{id}_X$. There is a homotopy ϕ between id_W and $h \circ h'$ by lemma 4.3 and the following equations.

$$\begin{aligned} (\text{id}_W, h \circ h') \circ (h + h) &= (h, h) = (h \circ \pi_1) \circ (c \times \text{id}_X) \\ h' \circ h \circ \pi_1 &= \pi_1 = (h' \circ \pi_1) \circ (\text{id}_{\Delta[1]} \times h) \\ h' \circ (\text{id}_W, h \circ h') &= (h', h') = (h' \circ \pi_1) \circ (c \times \text{id}_W) \end{aligned}$$

$$\begin{array}{ccccc}
X + X & \xrightarrow{h+h} & W + W & \xrightarrow{(id, h \circ h')} & W \\
c \times id \downarrow & & c \times id \downarrow & \nearrow \phi & \downarrow h' \\
\Delta[1] \times X & \xrightarrow{id \times h} & \Delta[1] \times W & \xrightarrow{h' \circ \pi_1} & X
\end{array}$$

$h \circ \pi_1$ (solid arrow from $\Delta[1] \times X$ to $W + W$)
 $h \circ \pi_1$ (dotted arrow from $\Delta[1] \times W$ to $W + W$)

Because $\phi \circ (id_{\Delta[1] \times h}) = h \circ \pi_1$, the cofibration h is weakly invertible by definition 5.2.

Since fibrations are closed under composition and since g is an acyclic fibration, $p_0 \circ g$ is a fibration if p_0 is.

Let $d : A \rightarrow B$ be a weakly invertible cofibration and let $a : A \rightarrow Y/f$ and $b : B \rightarrow Y$ satisfy $p_0 \circ a = b \circ d$. There is a morphism $c_0 : B \rightarrow X$ such that $c_0 \circ d = p_1 \circ a$ because X is fibrant. Let e_0 be one of the horn inclusions $1 \rightarrow \Delta[1]$. There is a morphism $c_1 : \Delta[1] \times B \rightarrow Y$ such that $c_1 \circ (id_{\Delta[1]} \times d)$ is the transpose of $p_0 \circ a$ and $c_0 \circ (e_0 \times id_B) = f \circ c_1$ because Y is fibrant and because of lemma 4.3.

$$\begin{array}{ccccc}
A & \xrightarrow{e_0 \times id} & \Delta[1] \times A & \xrightarrow{a^t} & Y \\
d \downarrow & & id \times d \downarrow & \nearrow c_1 & \downarrow ! \\
B & \xrightarrow{e_0 \times id} & \Delta[1] \times B & \xrightarrow{!} & 1
\end{array}$$

$f \circ c_0$ (solid arrow from B to $\Delta[1] \times A$)
 c_1 (dotted arrow from $\Delta[1] \times B$ to Y)

The transpose $c_1^t : B \rightarrow Y^{\Delta[1]}$ of c_1 and c_0 together factor as a morphism $c : B \rightarrow Y/f$ which is a filler for $p_0 \circ a = b \circ d$. By generalization, p_0 has the right lifting property with respect to all acyclic cofibrations and therefore is a fibration. \blacksquare

6. Descent

This purpose of this section is to replace arguments based on *minimal fibrations* in simplicial homotopy theory, in particular those that are related to homotopy type theory. We assume that there is a class M of *modest fibrations* in $\mathfrak{A}^{\Delta^{op}}$ with the following properties.

- M is closed under pullbacks along arbitrary morphisms.
- M is closed under composition.
- M is closed under fibred exponentiation. This means that if $f : X \rightarrow Z$ belongs to M and $g : Y \rightarrow Z$ is an arbitrary morphism, the fibred exponential $f_Z^g : (\Pi z : Z \mapsto X_z^{Y_z}) \rightarrow Z$ is in M too.

This requirement may give us more limits than we need, but the intended model of modest morphisms in \mathbf{Asm}_{ex} satisfies it.

- M contains all *regular* monomorphisms—a monomorphism is *regular* if it is an equalizer.

The interesting case is where M has the following structure in addition to the properties above.

6.1. DEFINITION. A universal modest fibration is a modest fibration $\text{uni} : U \rightarrow V$ such that every modest fibration $f : X \rightarrow Y$ is the pullback of uni along a unique morphism $\chi(f) : Y \rightarrow V$.

The following fact makes the universal modest fibration a potential model of homotopy type theory.

6.2. THEOREM. If $\text{uni} : U \rightarrow V$ is a universal modest fibration, then V is fibrant.

PROOF. For each horn inclusion $h : \Lambda_k[n] \rightarrow \Delta[n]$ and each pair of $v : \Lambda_k[n] \rightarrow V$, the modest fibration $v^*(\text{uni}) : \Lambda_k[n] \times_V U \rightarrow \Lambda_k[n]$ descends along h to form a modest fibration $D(v^*(\text{uni})) : \bullet \rightarrow \Delta[n]$ by lemma 6.3. There is a $\chi(D(v^*(\text{uni}))) : \Delta[n] \rightarrow V$ and $\chi(h_*(v^*(\text{uni}))) \circ h$ equals v by definition 6.1. ■

6.3. LEMMA. [Descent] For each horn inclusion $h : \Lambda_k[n] \rightarrow \Delta$ and each fibration $f : X \rightarrow \Lambda_k[n]$ there is a fibration $Df : Y \rightarrow \Delta[n]$ such that f is the pullback of Df along h .

PROOF. There is no horn inclusion for $n = 0$. In the case $n = 1$, there are two maps $1 \rightarrow \Delta[1]$. In both case $Df = \pi_0 : \Delta[1] \times X \rightarrow \Delta[1]$ suffices. For all the cases where $n > 0$ we use the following construction.

The functor D is defined (see definition 6.6) to have the following relation with a functor $K : \Delta/[n] \rightarrow \mathfrak{A}^{\Delta^{\text{op}}}/\Lambda_k[n]$.

$$\mathfrak{A}^{\Delta^{\text{op}}}/\Delta[n](\Delta(\xi), Df) \simeq \mathfrak{A}^{\Delta^{\text{op}}}/\Lambda_k[n](K(\xi), f)$$

This is possible because $\mathfrak{A}^{\Delta^{\text{op}}}/\Delta[n]$ is equivalent to the category of presheaves over $\Delta/[n]$ and the Yoneda lemma applies to those presheaves. The relation of D and K allows us to reduce the problem of lifting a horn against Df to the problem of lifting some finite colimit of objects in the image of K against f . These finite colimits have the required left lifting property by lemma 6.8. ■

The rest of this section works out the definition and the properties of K .

6.4. DEFINITION. The following defines the functor $K : \Delta/[n] \rightarrow \mathfrak{A}^{\Delta^{\text{op}}}/\Lambda_k[n]$.

1. A function $\xi : [m] \rightarrow [n]$ cuts $[m]$ into $n + 1$ posets $\xi_j = \{i : [m] \mid \xi(i) = j\}$.
2. Let $\|\xi\|$ be the number of elements of the product $\Pi i : ([n] - \{k\}) \mapsto \xi_i$.
3. Define $K_0(\xi) : [m + \|\xi\|] \rightarrow [n]$ as follows.

$$K_0(\xi)(i) = \begin{cases} \xi(i) & \xi(i) < k \\ k & \xi(i - \|\xi\|) \leq k \leq \xi(i) \\ \xi(i - \|\xi\|) & k < \xi(i - \|\xi\|) \end{cases}$$

4. In the lexicographical product $\Pi i : ([n] - \{k\}) \mapsto \xi_i$ tuples get the lexicographical ordering. This ordering determines priority by comparing elements in sequence.

$$(x_0, x_1, \dots) \leq_{\text{lex}} (y_0, y_1, \dots) \iff x_0 \leq y_0 \wedge (x_0 = y_0 \rightarrow (x_1, \dots) \leq_{\text{lex}} (y_1, \dots))$$

5. Let $\kappa(\xi) : \Pi i : ([n] - \{k\}) \mapsto \xi_i \rightarrow [m + \|\xi\|]$ be the nondecreasing injection κ which sends the lexicographical product $\Pi i : ([n] - \{k\}) \mapsto \xi_i$ to the interval in $[m + \|\xi\|]$ which starts at the least i such that $K_0(\xi)(i) = k$.
6. Let $\lambda(\xi) : [m] \rightarrow [m + \|\xi\|]$ be the unique nondecreasing injection which skips the image of κ . This means $\lambda(i) = i$ if $\xi(i) < k$ and $\lambda(i) = i + \|\xi\|$ if $\xi(i) \geq k$. Moreover $K_0(\xi) \circ \lambda(\xi) = \xi$.
7. For each morphism $\phi : \xi \rightarrow \xi'$ in $\Delta/[n]$ let $\phi_i : \xi_i \rightarrow \xi'_i$ be the fibrewise morphism and let $\Pi i : ([n] - \{k\}) \mapsto \phi_i$ be the corresponding map of the (lexicographical) products $\Pi i : ([n] - \{k\}) \mapsto \xi_i \rightarrow \Pi i : ([n] - \{k\}) \mapsto \xi'_i$.
8. Let $K_1(\phi) : K_0(\xi) \rightarrow K_0(\xi')$ be the unique nondecreasing function which satisfies $K_1(\phi) \circ \kappa(\xi) = \kappa(\xi') \circ \Pi i : ([n] - \{k\}) \mapsto \phi_i$ and $K_1(\phi) \circ \lambda(\xi) = \lambda(\xi') \circ \phi$.

The maps (K_0, K_1) define an endofunctor of $\Delta/[n]$. To get the functor, let $K(\xi) = h^*(\Delta(K_0(\xi)))$ for objects ξ of $\Delta/[n]$ and $K(\phi) = h^*(\Delta(K_1(\phi)))$ for morphisms, where h^* is the reindexing functor $\mathfrak{A}^{\Delta^{\text{op}}}/\Delta[n] \rightarrow \mathfrak{A}^{\Delta^{\text{op}}}/\Lambda_k[n]$ along the horn inclusion $h : \Lambda_k[n] \rightarrow \Delta[n]$. Concretely, K satisfies the following equations for all objects ξ of $\Delta/[n]$, all $x : \mathbf{U}(\text{dom}(K(\xi)))$ and all arrows ϕ of $\Delta/[n]$.

$$\begin{aligned}
\mathbf{U}(\text{dom}(K(\xi))) &= \{x : \mathbf{U}\Delta[m + \|\xi\|] K_0(\xi) \circ x : \mathbf{U}\Lambda_k[n]\} \\
K(\xi)(x) &= K_0(\xi) \circ x \\
\text{dim}(x) &= \text{dom}(x) \\
x \cdot \phi &= x \circ \phi \\
K(\phi)(x) &= K_1(\phi) \circ x
\end{aligned}$$

Thanks to the following property, the natural equivalence of homsets above extends to horn inclusions.

6.5. LEMMA. An intersection is a pullback square of monomorphisms. The functor K preserves all intersections of $\Delta/[n]$.

PROOF. The easiest way to see that is to describe the monomorphisms in the intersection as decidable predicates on the domain of a map $\xi : [m] \rightarrow [n]$. Maps $[m] \rightarrow \text{bool}$ that equals **true** at least once characterize monomorphisms to ξ . The action of the endofunctor (K_0, K_1) on the monomorphism has a parallel on predicates. For each $p : [m] \rightarrow \text{bool}$ the following equations define $p^K : [m + \|\xi\|] \rightarrow \text{bool}$.

$$p^K(\kappa(\vec{x})) = \forall i : [m] - \{k\} . p(x_i) \qquad p^K(\lambda(x)) = p(x)$$

It all works out because $\forall i : [m] - \{k\} . p(x_i) \wedge q(x_i)$ is equivalent to $(\forall i : [m] - \{k\} . p(x_i)) \wedge (\forall i : [m] - \{k\} . q(x_i))$. ■

6.6. DEFINITION. For each $f : X \rightarrow \Lambda_k[n]$ the functor $D : \mathfrak{A}^{\Delta^{\text{op}}} / \Lambda_k[n] \rightarrow \mathfrak{A}^{\Delta^{\text{op}}} / \Delta[n]$ satisfies the following equations.

$$\begin{aligned} \mathbf{U}(\text{dom}(D(f))) &= \sum_{\xi : \mathbf{U}\Delta[n]} (\mathfrak{A}^{\Delta^{\text{op}}} / \Lambda_k[n])(K(\xi), f) \\ \dim(\xi, x) &= \text{dom}\xi \\ (\xi, x) \cdot \phi &= (\xi \circ \phi, x \circ K(\phi)) \\ D(f)(\xi, x) &= \xi \end{aligned}$$

On morphisms in $\mathfrak{A}^{\Delta^{\text{op}}} / \Lambda_k[n]$, the functor D acts by composition: $Dm(\xi, x) = (\xi, m \circ x)$.

If \mathfrak{A} has infinite colimits, the functor D has a left adjoint K' that satisfies $K'\Delta \simeq K$. We cannot rely on that property here. The functor K does extend to finite colimits of simplices. Preservation of intersections implies that K also applies to finite unions of simplices like $\Lambda_l[m]$. The reason D preserves fibrations and their right lifting property, is that its partial left adjoint preserves the left lifting property.

6.7. DEFINITION. [Face notation] We introduce a notation for monomorphisms in $\Delta/[n]$ or more accurately for the result of applying Δ to them. For $\xi : [m] \rightarrow [n]$ and $\phi : [m] \rightarrow \mathbf{bool}$ let $\text{face}(\xi, p) : \text{dom}(\text{face}(\xi, p)) \rightarrow \Delta[m]$ satisfy

$$\begin{aligned} \mathbf{U}(\text{dom}(\text{face}(\xi, p))) &= \{\alpha : \mathbf{U}(\Delta[m]) \mid \forall i : \text{dom}(\alpha).p \circ \alpha(i) = \mathbf{true}\} \\ \dim(\alpha) &= \text{dom}(\alpha) \\ \alpha \cdot \chi &= \alpha \cdot \chi \\ \text{face}(\xi, p)(\alpha) &= \alpha \end{aligned}$$

$$\begin{array}{ccc} \bullet & \xrightarrow{\text{face}(\xi, p)} & \Delta[m] \\ & \searrow & \swarrow \Delta(\xi) \\ & \Delta[n] & \end{array}$$

With the help of this notation, we describe commutative triangles of monomorphisms as follows.

$$\Lambda_l[m] \xrightarrow{\hat{h}} \Delta[m] \xrightarrow{\Delta(\xi)} \Delta[n] = \bigvee_{i:[m]-\{l\}} \text{face}(\xi, \lambda x \mapsto x \neq i) \quad (1)$$

$$K(\Lambda_l[m]) \xrightarrow{K(\hat{h})} \Delta[m + \|\xi\|] \xrightarrow{\Delta(K_0(\xi))} \Delta[n] = \bigvee_{j:[m]-\{l\}} \text{face}(K_0(\xi), v_j) \quad (2)$$

$$h^*(K(\Lambda_l[m])) \xrightarrow{h^*(K(\hat{h}))} h^*(\Delta[m + \|\xi\|]) \xrightarrow{h^*(\Delta(K_0(\xi)))} \Lambda_l[n] = \quad (3)$$

$$\left(\bigvee_{i:[n]-\{k\}} \text{face}(K_0(\xi), u_i) \right) \wedge \left(\bigvee_{j:[m]-\{l\}} \text{face}(K_0(\xi), v_j) \right) = \quad (4)$$

$$\left(\bigvee_{\substack{i:[n]-\{k\} \\ j:[m]-\{l\}}} \text{face}(K_0(\xi), u_i \wedge v_j) \right) \quad (5)$$

$$h^*(\Delta[m + \|\xi\|]) \xrightarrow{h} \Delta[m + \|\xi\|] \xrightarrow{K_0(\xi)} \Delta[n] = \left(\bigvee_{i:[n]-\{k\}} \text{face}(K_0(\xi), u_i) \right) \quad (6)$$

where

$$\begin{aligned} u_i(x) &= (K_0\xi(x) \neq i) & v_i(\kappa(\xi)(\vec{x})) &= (\xi(i) = k \vee x_{\xi(i)} \neq i) \\ v_i(\lambda(\xi)(x)) &= (x \neq i) & u_i \wedge v_i(x) &= u_i(x) \wedge v_i(x) \end{aligned}$$

6.8. LEMMA. *The monomorphism $h^*K(\hat{h})$:*

$$\left(\bigvee_{\substack{i:[n]-\{k\} \\ j:[m]-\{l\}}} \text{face}(K_0(\xi), u_i \wedge v_j) \right) \rightarrow \left(\bigvee_{i:[n]-\{k\}} \text{face}(K_0(\xi), u_i) \right)$$

has the left lifting property with respect to all fibrations.

PROOF. The class of morphisms with the left lifting property is closed under compositions and pushouts. The monomorphism $h^*K(\hat{h})$ belongs to this class because of this closure property.

Decompose $h^*K(\hat{h})$ as the inclusions of the following three subobjects.

$$\begin{aligned} A &= \text{dom}(h^*K(\hat{h})) = \left(\bigvee_{\substack{i:[n]-\{k\} \\ j:[m]-\{l\}}} \text{face}(K_0(\xi), u_i \wedge v_j) \right) \\ B &= \left(\bigvee_{j:[m]-\{l\}} \text{face}(K_0(\xi), u_{\xi(l)} \wedge v_j) \right) \vee \left(\bigvee_{i:[n]-\{k, \xi(l)\}} \text{face}(K_0(\xi), u_i) \right) \\ C &= \text{cod}(h^*K(\hat{h})) = \left(\bigvee_{i:[n]-\{k\}} \text{face}(K_0(\xi), u_i) \right) \end{aligned}$$

Decompose $A \rightarrow B$ as a series of inclusions $A_j \rightarrow A_{j+1}$ that satisfy $A_0 = A$ and $A_{n+1} = B$. For $j : [n+1]$ define A_j as follows.

$$A_j = A \vee \left(\bigvee_{i:[j]-\{k, j, \xi(l)\}} \text{face}(K_0(\xi), u_i) \right)$$

Each inclusion $A_j \rightarrow A_{j+1}$ is a pushout of the inclusion of $A_j \cap \text{face}(K_0(\xi), u_{j+1})$ into $\text{face}(K_0(\xi), u_{j+1})$, where

$$\begin{aligned} A_j \cap \text{face}(K_0(\xi), u_{j+1}) &= \left(\bigvee_{i:[m]-\{l\}} \text{face}(K_0(\xi), u_{j+1} \wedge v_i) \right) \vee \\ &\quad \left(\bigvee_{i:[j+1]-\{k, \xi(l)\}} \text{face}(K_0(\xi), u_i \wedge u_{j+1}) \right) \end{aligned}$$

The faces $\text{face}(K_0(\xi), u_{j+1} \wedge v_i)$ and $\text{face}(K_0(\xi), u_i \wedge u_{j+1})$ all contain the point $\lambda(l)$ of $\Delta[m + \|\xi\|]$ which means that lemma 6.9 applies to the inclusion $A_j \cap \text{face}(K_0(\xi), u_{j+1}) \rightarrow \text{face}(K_0(\xi), u_{j+1})$. Hence $A \rightarrow B$ has the left lifting property.

If $\xi(l) = k$, then $B = C$ and the proof is complete. Suppose $\xi(l) = i \neq k$. First define the following maps $[m + \|\xi\|] \rightarrow \mathbf{bool}$ for each $\vec{p} : \Pi i : [n] - \{k\} \mapsto \xi_i$.

$$w_{\vec{p}}(\kappa(\vec{q})) = \mathbf{true} \quad w_{\vec{p}}(\lambda(q)) = ((\xi(q) = k \vee q \leq p_{\xi(q)}) \wedge \xi(q) \neq \xi(l))$$

Decompose $B \rightarrow C$ as a series of inclusions $B_j \rightarrow B_{j+1}$ that satisfy $B_0 = B$ and $B_{m+\|\xi\|+1} = C$. For $j : [m + \|\xi\| + 1]$ define B_j as follows.

$$B_j = B \vee \left(\bigvee_{\kappa(\vec{p}) < j} \text{face}(K_0(\xi), u_{\xi(l)} \wedge w_{\vec{p}}) \right)$$

When $j \neq \kappa(\vec{p})$ for any $\vec{p} : \Pi i : [n] - \{k\} \mapsto \xi_i$, the inclusion $B_j \rightarrow B_{j+1}$ is the identity arrow, which satisfies the left lifting property trivially.

Suppose $j \neq \kappa(\vec{p})$. Each inclusion $B_j \rightarrow B_{j+1}$ is a pushout of the inclusion of $B_j \cap \text{face}(K_0(\xi), u_{\xi(l)} \wedge w_{\vec{p}})$ into $\text{face}(K_0(\xi), u_{\xi(l)} \wedge w_{\vec{p}})$, where

$$\begin{aligned} B_j \cap \text{face}(K_0(\xi), w_{\vec{p}}) &= \bigvee_{i:[n]-\{\xi(l),k\}} \text{face}(K_0\xi, u_i \wedge w_{\vec{p}}) \vee \\ &\quad \bigvee_{i:[m]-\{l\}} \text{face}(K_0\xi, v_i \wedge w_{\vec{p}}) \vee \\ &\quad \bigvee_{\kappa(\vec{q}) < j} \text{face}(K_0\xi, w_{\vec{q}} \wedge w_j) \end{aligned}$$

Let $\vec{p}[l] : \Pi i : [n] - \{k\} \mapsto \xi_i$ satisfy $p[l]_j = l$ if $\xi(l) = j$ and $p[l]_i = p_i$ otherwise. The point $\kappa(\vec{p}[l])$ is a member of $\text{face}(K_0\xi, u_i \wedge w_{\vec{p}})$ for all $i : [n] - \{\xi(l), k\}$. It is also a member of $\text{face}(K_0\xi, w_{\vec{q}} \wedge w_{\vec{p}})$ for all $\vec{q} : \Pi i : [n] - \{k\} \mapsto \xi_i$ such that $\kappa(\vec{q}) < \kappa(\vec{p})$. When $\kappa(\vec{p}[l])$ is not a member of $\text{face}(K_0\xi, v_i \wedge w_{\vec{p}})$ for some i , then $\text{face}(K_0\xi, v_i \wedge w_{\vec{p}})$ is a subobject of a face of $B_j \cap \text{face}(K_0(\xi), \wedge w_{\vec{p}})$ that does contain $\kappa(\vec{p}[l])$.

Let $i : [m] - \{l\}$. If not $v_i(\kappa(\vec{p}[l]))$ then by definition $\xi(i) \neq k$ and $(p[l]_{\xi}(i) = i)$. If i is the least member of $\xi_{\xi(i)}$, then $\text{face}(K_0\xi, v_i \wedge w_{\vec{p}}) \subseteq \text{face}(K_0\xi, u_{\xi(i)} \wedge w_{\vec{p}})$. If $\xi(i-1) = \xi(i)$, then $\kappa(\vec{p}[i-1]) < \kappa(\vec{p})$ and $\text{face}(K_0\xi, v_i \wedge w_{\vec{p}}) \subseteq \text{face}(K_0\xi, W_{\vec{p}[i-1]} \wedge w_{\vec{p}})$.

Since $B_j \cap \text{face}(K_0(\xi), \wedge w_{\vec{p}})$ is a unions of faces that have the point $\vec{p}[l]$ in common, lemma 6.9 applies to the inclusion $B_j \cap \text{face}(K_0(\xi), \wedge w_{\vec{p}}) \rightarrow \text{face}(K_0(\xi), \wedge w_{\vec{p}})$. Therefore $B_j \rightarrow B_{j+1}$, $B \rightarrow C$ and $h^*K(\hat{h}) : A \rightarrow C$ all have the left lifting property. ■

6.9. LEMMA. [Face completion] *Let F be an inhabited decidable set of faces of $\Delta[p]$ which all have a point e of $\Delta[p]$ in common. The inclusion $\bigcup F \rightarrow \Delta[p]$ has the left lifting property with respect to all fibrations.*

PROOF. For all $j : [p]$ let F_j be the union of F with the set of j -dimensional faces of $\Delta[p]$ which contain the edge e . Because F is inhabited, $\bigcup F$ contains e and therefore $F_0 = F$. Because $\Delta[p]$ is a p -dimensional face of $\Delta[p]$ which contains e , $\bigcup F_p = \Delta[p]$. For $j > 0$ let S_j be the set of j -dimensional faces of $\bigcup F_j$ which are not already contained in $\bigcup F_{j-1}$. If a j -dimensional face $\text{face}(\Sigma)$ of $\bigcup F_j$ opposes e , it is part of a higher dimensional face which is a member of F . Therefore each face $\text{face}(\Sigma) : S_j$ contains e . For this reason $\text{face}(\Sigma) \cap \bigcup F_{j-1}$ is the horn whose central edge is e . The inclusion $\bigcup F_{j-1} \rightarrow \bigcup F_j$ is therefore the pushout of a coproduct of horn inclusions indexed over S_j . Therefore the inclusion has the left lifting property. Because composition preserves the left lifting property, $\bigcup F = F_0 \rightarrow F_p = \Delta[p]$ has it too. ■

References

- [1] Aurelio Carboni and E. M. Vitale. “Regular and exact completions”. In: *J. Pure Appl. Algebra* 125.1-3 (1998), pp. 79–116. ISSN: 0022-4049. DOI: 10.1016/S0022-4049(96)00115-6. URL: [http://dx.doi.org/10.1016/S0022-4049\(96\)00115-6](http://dx.doi.org/10.1016/S0022-4049(96)00115-6).

- [2] Paul G. Goerss and John F. Jardine. *Simplicial Homotopy Theory*. Vol. 174. Progress in Mathematics. Basel: Birkhäuser, 1999. ISBN: 978-3-0348-8707-6.
- [3] Mark Hovey. *Model Categories*. Vol. 63. Mathematical Surveys and Monographs. Providence: American Mathematical Society, 1999. ISBN: 978-0-8218-4361-1.
- [4] J. M. E. Hyland, E. P. Robinson, and Guiseppe Rosolini. “The discrete objects in the effective topos”. In: *Proc. London Math. Soc. (3)* 60.1 (1990), pp. 1–36. ISSN: 0024-6115. DOI: 10.1112/plms/s3-60.1.1. URL: <http://dx.doi.org/10.1112/plms/s3-60.1.1>.
- [5] Chris Kapulkin, Peter Lumsdaine, and Vladimir Voevodsky. “The Simplicial Model of Univalent Foundations”. In: *preprint* (2012). arXiv:1211.2851 [math.1o].
- [6] Saunders Mac Lane and Ieke Moerdijk. *Sheaves in geometry and logic*. Universitext. A first introduction to topos theory, Corrected reprint of the 1992 edition. New York: Springer-Verlag, 1994, pp. xii+629. ISBN: 0-387-97710-4.
- [7] Jaap van Oosten. *Realizability: an introduction to its categorical side*. Vol. 152. Studies in Logic and the Foundations of Mathematics. Elsevier B. V., Amsterdam, 2008, pp. xvi+310. ISBN: 978-0-444-51584-1.
- [8] Giuseppe Rosolini. “About modest sets”. In: *Internat. J. Found. Comput. Sci.* 1.3 (1990). Third Italian Conference on Theoretical Computer Science (Mantova, 1990), pp. 341–353. ISSN: 0129-0541. DOI: 10.1142/S0129054190000242. URL: <http://dx.doi.org/10.1142>

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